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$$\boxed{4\left(\frac{\partial^2}{\partial z\bar{z}}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}$$

Harmonic function's

Let $u(x,y)$ be a function of two real variables x and y defined in a region D. $u(x,y)$ is said to be a harmonic function if

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$ and this equation is called Laplace equation.

Theorem:

The real and Imaginary parts of an analytic function are harmonic functions

Proof:

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function

Then u and v have continuous partial derivaties of first order which satisfy the CR equations given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Further

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is a Harmonic function.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v is a Harmonic function.

Remark: Laplance's equation provides a necessary condition for a function's to be the real (or) imaginary parts analytic function.

Example: $u(x,y) = x^2 - y^2$, we have

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial^2 u}{\partial x^2} =$ $\quad \frac{\partial u}{\partial x} = 2y$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2+2 = 4 \neq 0.$$

$\therefore$ u is not harmonic function and hence it cannot be the real part of any analytic function.

Definition: Conjugate Harmonic function:

Let $f = u + iv$ be an analytic function in a region D. The v is said to be a conjugate harmonic function of u.

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Theorem:

Let $f = u + iv$ be an analytic function in a region D. Then v is harmonic conjugate of $v \Leftrightarrow u$ is a harmonic conjugate of $-v$.

Proof:

Let v be a harmonic conjugate of u then $f = u + iv$ is analytic

$\Leftrightarrow$ if $-iu - v$ is also analytic

$\Leftrightarrow$ if u is a harmonic conjugate of $-v$.

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Theorem:

Any two harmonic conjugate of a given harmonic function u in a region D differ by a real constant.

Proof:

Let u be a harmonic function

Let v and v^* be two harmonic conjugates of u. $u+iv$ and $u+iv^*$ are analytic in D hence by the C-R equation we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v^*}{\partial x}$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial v^*}{\partial x}$$

Hence $\frac{\partial}{\partial y}(v - v^*) = 0$ and $\frac{\partial}{\partial x}(v - v^*) = 0$

$v - v^* = c$

$v = v^* + c$

where c is real constant.

5 Mark Milne - Thompson Method.

C.I.F. proof: Let $u(x,y)$ be a given harmonic function. Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

Then $f'(z) = u_x(x,y) + iv_x(x,y)$

$= u_x(x,y) - iu_y(x,y)$

Let $\phi_1(x,y) = u_x(x,y)$ and $\phi_2(x,y) = u_y(x,y)$

We have $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

$z = x+iy$, $\bar{z} = x-iy$

Hence $f'(z) = \phi_1\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - i\phi_2 \ldots$

$\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$ putting $z=\bar{z}$ we obtain $f'(z) = \phi_1(z,0) - i\phi_2(z,0)$

Hence $f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$

Note:

It can be proved in a similar way that the analytic function $f(z)$ with a given harmonic $v(x,y)$ as imaginary part is given by.

$$f(z) = \int [\psi_1(z,0) + \psi_2(z,0)]\, dz + c$$

where $\psi_1(x,y) = v_y$ and $\psi(x,y) = v_x$.

problem: 1.

S.T $u(x,y) = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmonic. Find an analytic function $f(z)$ interms of z with the given u for its real part

Sol

$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2$

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$

$\therefore u_{xx} + u_{yy} = 0$.

Hence u is harmonic.

Now, Let $\phi_1(x,y) = u_x$ and $\phi_2(x,y) = u_y$

$\therefore \phi_1(z,0) = \cos z \cosh(0) - 2\sin z \sinh 0 + 2z$

$= \cos z + 2z$

$\text{|||}^{ly}\ \phi_2(z,0) = 2\cos z + 4z$.

$\therefore f(z) = \int \left(\phi_1(z,0) - i\phi_2(z,0) \right) dz$

(by Milne Thomson Method)

$= \int \left[\cos z + 2z - i\,(2\cos z + 4z) \right] dz$

$f(z) = \sin z + z^2 - 2i\sin z - 2iz^2 + c.$

$f(z) = (\sin z + z^2) + i(-2\sin z - 2z^2) + c$

2. If $f(z) = u(x,y) + iv(x,y)$ is analytic function and $u(x,y) = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ find $f(z)$.

Sol

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz$

[by milne Thomson method]

Let $\phi_1(x,y) = u_x$ $\quad \phi_2(x,y) = u_y$

$$u_x = \frac{(\cosh 2y + \cos 2x)\, 2\cos 2x + \sin 2x \, 2\sin 2x}{(\cosh 2y + \cos 2x)^2}$$

$$u_x = \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

$$\phi_1(z,0) = u_x = \frac{2\cosh(0)\cos 2z + 2}{(\cosh 0 + \cos 2z)^2}$$

$$= \frac{2\cos 2z + 2}{(1 + \cos 2z)^2}$$

$$= \frac{2(1 + \cos 2z)}{(1 + \cos 2z)^2} = \frac{2}{1 + \cos 2z}$$

$= \frac{2}{1+\cos 2z} = \frac{2}{2\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z.$

$\phi_1(z,0) = u_x = \sec^2 z.$

$$u_y = \frac{(\cosh 2y + \cos 2x)(0) + \sin 2x(\sinh 2y \cdot 2)}{(\cosh 2y + \cos 2x)^2}$$

$u_x = \sec^2 z = \frac{0 - 2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$

$u_y = 0$

$$\phi_2(z,0) = \frac{-2\sin 2z \sinh z(0)}{(\cosh 2(0) + \cos 2z)^2}$$

$$= \frac{0}{\cosh(2 0) + \cos 2(z)^2} = 0.$$

$f(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz + c$

$= \int \sec^2 z - i(0)\, dz$

$= \int \sec^2 z\, dz$

$= \tan z + c$

$\boxed{f(z) = \tan z + c}$

3. P.T. $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

$u = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2$

$u_{xx} = -6x$

$u_y = 6xy$

$u_{yy} = 6x$

$u_{xx} + u_{yy} = 0$ hence u harmonic

Let v be a harmonic conjugate of u

$\therefore f(z) = u + iv$ is analytic

By C-R equation's we have

$v_y = u_x = 2 - 3x^2 + 3y^2$

$\therefore$ Integrating with respect to y

we get

$v = 2y - 3x^2y + y^3 + \lambda(x) \rightarrow (1)$

where $\lambda(x)$ is an arbitrary func of

$v_x = -6xy + \lambda'(x)$

Now, $v_x = -u_y$ gives $-6xy + \lambda'(x) =$

then $\lambda'(x) = 0$ so that $\lambda(x) = 0$

(where c is constant)

Thus $v = 2y - 3x^2y + y^3 + c$ (from (1))

Now $f(z) = (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3)$

$+ ic$

$= 2(x + iy) - [(x^3 - 3xy^2) + i(3x^2y - y^3)] + ic$

$= 2z - z^3 + ic$

$\therefore f(z) = 2z - z^3 + ic$ is the required analytic function.

4. S.T $u = \log\sqrt{x^2+y^2}$ is harmonic and hence determine is conjugate and hence find the corresponding analytic function $f(z)$.

Sol $u = \log\sqrt{x^2+y^2} = \frac{1}{2}(\log(x^2+y^2))$

$\therefore u_x = \frac{x}{x^2+y^2}$ $u_{xx} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2}$

III^{ly} $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

obviously $u_{xx}+v_{yy}=0$ and hence u is harmonic

Let v be a harmonic conjugate of u $\therefore f(z) = u+iv$ is analytic

By C-R equations we have

$v_y = u_x = \frac{x}{x^2+y^2}$

$\therefore$ Integrating w.r.t y we get

$v = \tan^{-1}(y/x) + \psi(x)$ where $\psi(x)$ is an arbitrary function.

Now $v_x = \frac{1}{1+y^2/x^2}\left(-y/x^2\right) + \psi^{-1}(x)$

Also $v_x = -u_y \Rightarrow \frac{-y}{x^2+y^2} + \phi(x) = \frac{-y}{x^2+y^2}$

So that $\phi'(x) = 0$

hence $\phi(x) = c$

$\therefore v = \tan^{-1}(y/x) + c$

$f(z) = u+iv$

$= \log\sqrt{x^2+y^2} + i(\tan^{-1}(y/x) + c$

5. Find the analytic function $f(z) = u + iv$ is $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

Sol $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x + v_x = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

and

$$u_y + v_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$\therefore$ The required function $f(z) = u + iv$ is to be analytic, u and v satisfy the C-R equations $u_x = v_y$ and $u_y = -v_x$ in (2) we get. Using these equations

$$u_x - u_y = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$\therefore u_x(z,0) - u_y(z,0) = \frac{2(1 - \cos 2z)(\cos 2z - 2\sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z.$$

using C-R equations in (3) we get

$$u_y + v_x = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$u_y(z,0) + u_x(z,0)$ — (5)

Now adding (4) & (5) we get

$2u_x(z,0) = -\operatorname{cosec}^2 z$.

$u_x(z,0) = -\frac{1}{2}\operatorname{cosec}^2 z \rightarrow (6)$

Subtration (4) from (5) we get,

$2u_y(z,0) = \operatorname{cosec}^2 z \rightarrow (7)$

Now $f(z) = u(z,0) + iv(z,0)$

$\Rightarrow f'(z) = u_x(z,0) + iv(z,0)$

$= u_x(z,0) - i.u_y(z,0)$

$= -\frac{1}{2}(1+i)\operatorname{cosec}^2 z$ (using (6) & (7))

Integrating w.r.t z we get

$f(z) = \left(\frac{1+i}{2}\right)\cot z + C$.

6. Given $v(x,y)$ $x^4 - 6x^2y^2 + y^4$ find $f(z)$ $= u(x,y) + iv(x,y)$ such that $f(z)$ is analytic.

Sol: It can be easily verified that $v(x,y)$ is harmonic and

Now $v_x = 4x^3 - 12xy^2$

$v_y = -12x^2y + 4y^3$

Let $f(z) = u + iv$ be the required analytic function.

By C.R equation $u_x = v_y$

$\therefore u_x = -12x^2y + 4y^3$

Integrating w.r.t x we get

$u = -4x^3y + 4xy^3 + \lambda(y)$.

where $c(y)$ is an arbitrary function of y

$$-4x^3 + 12xy^2 + \lambda'(y) = -U_x$$

$$U_y = -(4x^3 - 12xy^2) = -4x^3 + 12xy^2 + \lambda'(y)$$

$\lambda'(y) = 0$ so that $\lambda(y) = c$ where c is a constant.

Thus $u = -4x^3y + 4xy^3 + c$.

$$f(z) = [-4x^3y + 4xy^3 + c] - i(x^4 - 6x^2y^2 + y^4)$$

$$= i(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) +$$

$$= i(x + iy)^4 + c$$

$$= iz^4 + c$$

Aliter (Milne Thompson Method)

Let $\psi_1(x,y) = V_y$ and $\psi_2(x,y) = V_x$

$$\psi_1(x,y) = -12x^2y + 4y^2 \text{ and}$$

$$\psi_2(x,y) = 4x^3 - 12xy^2$$

$\psi_1(z,0) = 0$ and $\psi_2(z,0) = 4z^3$

$$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)] \, dz$$

$$= i \int 4z^3 \, dz$$

$$\boxed{f(z) = iz^4 + c.}$$

(7) Find the analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin$

Sol

$$u - v = e^x(\cos y - \sin y) \rightarrow (1)$$

$$u_x - v_x = e^x(\cos y - \sin y) \rightarrow (2)$$

and

$$u_y - v_y = -e^x(\sin y + \cos y) \rightarrow (3)$$

Since the required function is to be analytic it has to satisfy the C-R equation using C-R equation in (3) we get

$$-v_x - u_x = -e^x(\sin y + \cos y) \rightarrow (4)$$

solving (2) and (4) we get

$-v_x$ $-u_x$

$$u_x = e^x \cos y \rightarrow (5)$$

$$v_y = e^x \sin y \rightarrow (6)$$

Integrating (6) p.w.r to 'x' we get

$$v_x = e^x \sin y + f(y)$$

$$v_y = e^x \cos y + f'(y) \rightarrow (7)$$

using C-R equation in (5) and (7) we get

$$f'(y) = 0$$

hence $f(y) = c_1$

where $\therefore$ $v = e^x \sin y + c_1$

from (1)

$$u = e^x \cos y + c_2$$

Now $f(z) = u + iv$

$$= e^x \cos y + ie^x \sin y + c_2 + ic_1$$

$$= e^x(\cos y + i\sin y) + (c_2 + ic_1)$$

$$= e^x e^{iy} + \alpha$$

$$\boxed{f(z) = e^z + \alpha}$$

8. If $u+v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$, find the analytic function $f(z)$ interms of z.

Sol $u+v = (x-y)(x^2+4xy+y^2) \rightarrow (1)$

Differentiate (1) p.w.r.t 'x'. we get

$u_x + v_x = (x^2+4xy+y^2) + (x-y)(2x+4y)$

Differentiat (1) p.w.r.t 'y' we get $\rightarrow (2)$

$u_y + v_y = -(x^2+4xy+y^2) + (x-y)(4x+2y) \rightarrow (3)$

Since $f = u+iv$ is analytic, u and v satisfy the C-R equation.

$u_x = v_y$

$u_y = -v_x$

Using C-R equation in (3) we get

$-v_x + u_x = -(x^2+4xy+y^2) + (x-y)(4x+2y) \rightarrow (4)$

Adding (2) & (4) we get

$2u_x = (x-y)(6x+6y)$

$u_x = 3(x^2-y^2)$

subracting (4) and (2) we get

$v_x = 6xy \rightarrow (5)$

using C-R equation in (5) we get

$u_y = -6xy \rightarrow (6)$

Let $\phi_1(x,y)$ u_x and

$\phi_2(x,y) = uy$

$\therefore \phi(z,0) = 3z^2$ and $\phi_2(z,0) = 0$

By Milne - Thompson method

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] dz$

$= \int 3z^2 dz$

$= \cancel{3} \frac{z^3}{\cancel{3}} + c$

$= z^3 + c$ //.

9. Find the real part of the analytic function whose imaginary part is $e^{-x} [2xy \cos y + (y^2 - x^2) \sin y]$.

Sol

Let $v = e^{-x} [2xy \cos y + (y^2 - x^2) \sin y]$ and $f(z) = u + iv$ be the required analytic function. We use Milne Thompson method to find the harmonic conjugate u of v

$v = 2e^{-x} xy \cos y + (y^2 - x^2) \sin y \, e^{-x}$

$\psi_1(x,y) = v_y = 2e^{-x} [x \cos y - xy \sin y] + [2y \sin y + (y^2 - x^2) \cos y] e^{-x}$

$\psi_2(x,y) = v_x = -2xy \cos y e^{-x} - (y^2 - x^2) \sin y e^{-x} + 2y \cos y e^{-x} - 2x \sin y e^{-x}$

$\psi_1(z,0) = 2e^{-z} [z] + [-z^2] e^{-z}$

$\psi_2[z,0] = 0$ //

WKT

$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)] dz + c$

$= \int (2z e^{-z} - z^2 e^{-z}) dz + c$

$f(z) = \int 2z e^{-z} dz - \int z^2 e^{-z} dz + c$

$$2\int z e^{-z}\,dz = 2\left(-ze^{-z}\right) + \int e^{-z}\,dz$$

$$= 2\left[-ze^{-z} + \int e^{-z}\,dz\right]$$

$$\int z^2 e^{-z}\,dz = -z^2 e^{-z} + 2\int z e^{-z}\,dz$$

$$= -z^2 e^{-z} + 2\left[-ze^{-z} - e^{-z}\right)$$

$$f(z) = -2ze^{-z} - 2e^{-z} + z^2 e^{-z} + 2ze^{-z} + 2e^{-z}$$

$$f(z) = z^2 e^{-z}$$

$$f(z) = (x+iy)^2 e^{-(x+iy)}$$

$$= (x^2 - y^2 + 2ixy)e^{-x}e^{-iy}$$

$$= x^2 e^{-x}\cos y - y^2 e^{-x}\cos y + 2ixye^{-x}$$

$$- i x^2 \sin y e^{-x} + i y^2 \sin y e^{-x} + 2xy \sin y$$

$$f(z) = e^{-x}\left[x^2\cos y - y^2\cos y + 2xy\sin y\right]$$

$$+ i\left[e^{-x}(2xy\cos y - x^2\sin y + y^2\sin y\right]$$

Real part of $u = e^{-x}(x^2\cos y - y^2\cos y + 2xy\sin y$

10. Find The constant 'a' so that $u(x,y) = ax^2 - y^2 + xy$ is harmonic find an analytic function $f(z)$ for which u is the real part also find is harmonic Conjugate.

Sol

$$u(x,y) = ax^2 - y^2 + xy$$

$u_x = 2ax + y$	$u_y = -2y + x$
$u_{xx} = 2a$	$u_{yy} = -2$

$\therefore$ a is harmonic

$u_{xx} + u_{yy} = 0$

$2a - 2 = 0$

$\Rightarrow a = 1$

$u = x^2 - y^2 + xy$

$\phi = u_x = 2x + y$

$\phi_1(z,0) = 2z$

$\phi_2(x,y) = u_y = -2y + x$

$\phi_2(z,0) = z$.

By milne thompson method

$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + k$

$f(z) = \int 2z\,dz - i\int z\,dz + c$

$f(z) = z^2 - i\frac{z^2}{2} + c$

$f(z) = (x+iy)^2 - \frac{i}{2}(x+iy)^2 + c$

$= x^2 - y^2 + 2ixy - \frac{i}{2}(x^2 - y^2 + 2ixy) + c$

$f(z) = [[x^2 - y^2 + xy] + 2xy) - i\frac{x^2}{2} + i\frac{y^2}{2} + c$

$\boxed{v = 2xy - \frac{x^2}{2} + \frac{y^2}{2}}$ //.

11. If $u(x,y)$ is a harmonic function in a region D, prove that $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in D.

Sol

Let $U = \frac{\partial u}{\partial x}$ and $V = \frac{\partial u}{\partial y}$

$\therefore f(z) = U + iV$ Since u is harmonic

U and V have continuous first order partial derivatives and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \longrightarrow (1)$

Also $\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$ (using (1))

hence $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ $\quad u_x=\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2}$

Now. $\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y \partial x}$

$=\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$

$=-\frac{\partial v}{\partial x}$

hence $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Thus the partial derivaties of U and V satisfy the C-R equations in D.
hence f is analytic in D.

12. If u and v are harmonic function satisfy the C-R equation, in region D. Then $f=u+iv$ is analytic in D.

Sol since u and v are harmonic the first order partial derivaties of u and v are continuous

Also u and v satisfy the C-R equation in D

Hence $f = u + iv$ is analytic in D.

13. prove that the real Imaginary part of an analytic function when expressed in polar form satisfies The equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ (Thus equation in equation is the laplace equation in polar form].

Sol

W.K.T CR equation in polar form are given by $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \rightarrow (2)$$

we diminate v form (1) and (2) Diff (1) partially w.r to r and (2) partially w.r.to. θ

$$\frac{\partial^2 v}{\partial r \partial \theta} = r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \rightarrow (3)$$

$$\frac{\partial^2 v}{\partial \theta \partial r} = -\frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} \rightarrow (4)$$

Sino $\frac{\partial^2 v}{\partial r \partial \theta} = \frac{\partial^2 v}{\partial \theta \partial r}$

be have $r \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r} \frac{\partial^2 u}{\partial \theta^2}$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = -\frac{1}{r} \frac{\partial^2 u}{\partial \theta^2}$$

III ly $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0$.

14. ϕ and ψ are function of x and y satisfying laplace's equation if $u = \phi y - \psi x$ and $v = \phi x + \psi y$ P.T $u+iv$ is analytic.

Sol

Given that ϕ and ψ satify laplace's equation.

Hence $\phi_{xx} + \phi_{yy} = 0 \rightarrow (1)$

and $\psi_{xx} + \psi_{yy} = 0 \rightarrow (2)$

also $u = \phi_y - \psi_x$ and $v = \phi_x + \psi_y$

hence $u_x = \phi_{xy} - \psi_{xx}$

$v_x = \phi_{xx} + \psi_{xy}$

$u_y = \phi_{xx} - \psi_{xy}$

$= -\phi_{yy} + \psi_{xy}$ (∵ by (1))

and $v_y = \phi_{yx} + \psi_{xx}$ (∵

$= \phi_{yx} - \psi_{xx}$ (∵ by (2))

Thus $v_x = u_y$ and $u_y = -v_x$

Since ϕ and ψ are harmonic all the partial derivative and continuous

Hence $u+iv$ is analytic.

15. S.T if u and v are conjugate harmonic function the product uv is a harmonic function

~~Since~~ Since u and v are conjugate harmonic function the product uv is a harmonic function. we have

$$u_{xx} + u_{yy} = 0 \rightarrow (1)$$

$$v_{xx} + v_{yy} = 0 \rightarrow (2)$$

$$u_x = v_y \rightarrow (3)$$

$$u_y = -v_x \rightarrow (4).$$

Now let $\phi = uv$

$$\phi_x = uv_x + vu_x$$

$$\phi_{xx} = uv_{xx} + 2u_x v_x + vu_{xx}.$$

$|||^{ly}$ $\phi_{yy} = uv_{yy} + 2u_y v_y + vu_{yy}$

$$= uv_{yy} + 2v_x u_x + vu_{yy}$$

[using 2 and (4)]

Now.

$$\phi_{xx} + \phi_{yy} = u(v_{xx} + v_{yy}) + v(u_{xx} + u_{yy})$$

$= 0$ (using (1) & (2))

$= uv$ is a harmonic function.

16. If $f(z)$ is analytic P.T

$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|$

Sol

Let $f(z) = u+iv$

$|f(z)|^2 = u^2+v^2 = \phi$ (say)

and $f'(z) = u_x + iv_x$.

$$\frac{\partial \phi}{\partial x} = 2uu_x + 2vv_x$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = 2[u_x^2 + uu_{xx} + v_x^2 + vv_{xx}$$

III^{ly} $$\frac{\partial^2 \phi}{\partial y^2} = 2[u_y^2 + uu_{yy} + v_y^2 + vv_{yy}$$

$$= 2[v_x^2 + uu_{yy} + u_x^2 + vv_{yy}$$

Since u and v are harmonic

$u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} =$

Adding (1) and (2) using (3) we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4(u_x^2 + v^2)$$

$$= 4|u_x + iv_x|^2$$

$$= 4|f'(z)|^2$$

$$= 4|f'(z)|^2$$

Hence the Result.

7. If $f(z) = u+iv$ is analytic and $f'(z) \neq 0$ P.T (i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log |f(z)| = 0$ (ii) $\nabla^2 \text{amp } f(z) = 0$.

Sol

$\log f(z) = \log |f(z)| + i \text{ amp } f(z)$

$\because f(z) \neq 0$ $\log |f(z)|$ exists

further since $f(z)$ is analytic and

$f(z) \neq 0$

$\log f(z)$ is also analytic

$\therefore$ $\log |f(z)|$ and amp $f(z)$ are

The real & imaginary parts of the analytic function of $\log f(z)$

Hence both $\log |f(z)|$ and amp $f(z)$ satisfy the Laplance equation.

(i) $\frac{\partial^2}{\partial x^2} (\log |f(z)|) + \frac{\partial^2}{\partial y^2} (\log |f(z)|) = 0$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (\log |f(z)|) = 0$

(ii) $\frac{\partial^2}{\partial x^2} (\text{amp } f(z)) + \frac{\partial^2}{\partial y^2} (\text{amp } f(z)) = 0$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (\text{amp } f(z)) = 0$

$\nabla^2 \text{ amp } f(z) = 0.$

18. Given the function $w = z^3$, $w = u + iv$ show that u and v satisfy the C-R equation P.T the families of curves $u = c_1$ $v = c_2$ (c_1 and c_2 are constant) are orthogonal to each other:

sol

$$w = z^3 = (x+iy)^3$$

$$w = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$$

$$u_x = 3x^2 - 3y^2 \qquad v_x = 6xy$$

$$u_y = -6xy \qquad v_y = 3x^2 - 3y^2$$

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy = -v_x$$

∴ u & v satisfy the C-R equation.

$$u_{xx} = 6x \qquad v_{xx} = 6y$$

$$u_{yy} = -6x \qquad v_{yy} = -6y$$

$$\therefore u_{xx} + u_{yy} = 0$$

$$v_{xx} + v_{yy} = 0$$

u and v satisfy laplace equation. Slope of the tangent at (x_0, y_0) for the curve $u = c_1$ is given by.

$$m = \frac{dy}{dx}.$$

$u = c_1 \Rightarrow x^3 - 3xy^2 = c_1$

Diff w.r.t 'x' $\Rightarrow 3x^2 - 3\left(2xy\frac{dy}{dx} + y^2\right) = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{3(x^3 - y^3)}{6xy} = \frac{x^2 - y^2}{2xy}$$

$$m_1 = \frac{x_0^2 - y_0^2}{2x_0 y_0}$$

Now $v = c_2$

$3x^2y - y^3 = c_2$

Diff w.r.to x

$$3\left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

$$2xy + x^2\frac{dy}{dx} - y^2\frac{dy}{dx} = 0$$

$$2xy + (x^2 - y^2)\frac{dy}{dx} = 0$$

$$(x^2 - y^2)\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx} = -\frac{2xy}{x^2 - y^2}.$$

Slope of the tangent at (x_0, y_0) for the $v = c_2$ is given by

$$m_2 = \frac{-2x_0 y_0}{x_0^2 - y_0^2}$$

$m_1 m_2 = -1$

$\therefore$ The two families of ~~curve~~ curve are orthogonal